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Truth and Hard Times

In the development of civilization eras recur in which the truth is not in good repute. The last of these reached its climax in the United States a scant two dozen years ago; and the next, if not already upon us, perilously impends. During such times an attitude of detachment is commonly suspected to be political disloyalty, and abstract observation has to submit to unfriendly misinterpretation.

It is no particular news, and not particularly laudable, that most sciences find it easy enough to trim their sails to the wild winds of such disasters. The current estate of science in Germany is a pointed instance of this. In general equivalent ends may be effected without such a violent turning of coats. A search for new vitamins may be quietly replaced by a search for lethal gases, and an interest in molecular structure may evolve into an interest in ballistics.

Mathematics is not a science which finds it easy to adapt itself to drastic changes in popular opinion. Only the fact that amateur espionage will rarely have the courage to attempt to decode it keeps it from eclipse. But when truth in general goes begging, those devoted to its mathematical forms have either to go begging with it, or to go begging for it. Of course more distinction attaches to going begging for something, than to going begging with something. There are committees to raise funds, programs to be formulated, and much useful bustling about to participate in agreeably. The group preferring to beg for mathematics will certainly include a large number of sincere men who believe that mathematics ought to influence world affairs. It will also include all the good mixers. But it will not include all mathematicians. There will be some old shell-backs who will choose to go begging with mathematics. When the world weather is foul, look for them in a dark corner bomb proofed with books, where, cloaked from the ankles to a level well above the ears in a thick monograph and holding a sheaf of research close enough before the eyes to cut off all glimpses of events, they nurse their wits beside the hearthfire of truth in symbolic form. The dawn of another day will thaw them out again.

N. E. Rutt,
Louisiana State University.

The Solution of the Trinomial Equation in Infinite Series by the Method of Iteration

By Newman A. Hall*
Queens College

The infinite series expressions for the roots of a trinomial equation may be obtained by the application of the Lagrange expansion theorem or by the solution of the differential equation satisfied by the roots, the so-called differential resolvents of the algebraic equation. The first results in this connection seem to be due to R. Harley [1]† (1862) who discussed the differential resolvents for the trinomial equation and exhibited the explicit solutions for equations of fifth degree and less. The Lagrange expansion theorem was first applied to this problem by P. Nekrassoff [1] (1883) who also duplicated the essentials of Harley's work. W. Heymann [1]-[6] considered by several methods the trinomial and general algebraic equations.

Subsequent investigators have built largely on the methods introduced by these three, extending them to cover the general equations and considering more rigorously the properties of the series. The differential resolvents have been the basis of the work of G. Belardinelli and Hj. Mellin, while R. Birkeland has developed the results more by use of the Lagrange theorem.

It is the purpose of this paper to present a new method for the development of these series solutions by extending the familiar method of numerical approximation by iteration, to the general trinomial equations. It would seem convenient to be able to proceed directly to a general result without the introduction of some essentially foreign theory.

The method of iteration as applied to the solution of algebraic or transcendental equations consists of reinterpreting the equation as a recurrence relation to proceed from some conveniently assumed zero approximation to the root by successively closer approximations. Thus, when the equation is written in the form:

$$z = f(z), \tag{1}$$

the recurrence formula

$$z_{k+1} = f(z_k) \tag{2}$$

*Presented to the American Mathematical Society, April 9, 1938.
†Numbers within square brackets refer to the bibliography.

will lead to a root of (1) if the zero approximation, z_0, is chosen so that the sequence z_k is convergent.

The trinomial equation can always be reduced by a direct change of variable to the form:

$$z^{m+n} - z^m + a = 0, \tag{3}$$

where m and n are relative prime, positive integers and a is some complex or real number.

The zero approximations to the iterations for the roots of (3) are conviently suggested by the behavior of these roots as $|a| \to \infty$ and as $|a| \to \infty$. It will be necessary to consider both cases in order to secure later successive approximations convergent to the roots for all values of a.

When $|a| \to \infty$, the roots in absolute value will approach either zero or one. For those approaching zero, the equation is approximately

$$-z^m + a = 0;$$

and for those approaching one,

$$z^n - 1 = 0.$$

Thus there are the $m+n$ zero approximations:

$$z_0 = \alpha, \qquad \alpha^n = 1, \tag{4.1}$$

$$z_0 = \beta a^{1/m}, \qquad \beta^m = 1, \tag{4.2}$$

corresponding to the $m+n$ roots of (3). Again, when $|a| \to \infty$ the absolute value of all the roots will become infinitely large, and the equation will have the approximate form:

$$z^{m+n} + a = 0;$$

giving the $m+n$ approximations:

$$z_0 = \gamma a^{1/m+n}, \qquad \gamma^{m+n} = -1. \tag{5}$$

To secure convergence, the recurrence formulæ corresponding to (2) to be derived from (3) must be correlated to the zero approximations (4) and (5). For (4.1), (3) is written:

$$z = \alpha(1 - az^{-m})^{1/n},$$

For (4.2), (3) is written:

$$z = \beta a^{1/m}(1 - z^n)^{-1/m}.$$

For (5), (3) is written:

$$z = \gamma a^{1/n+m}(1 - a^{-1}z^m)^{1/n+m}.$$

Therefore the respective iteration formulæ are:

$$z_{k+1} = \alpha(1 - a z_k^{-m})^{1/n}, \tag{6.1}$$

$$z_{k+1} = \beta a^{1/m}(1 - z_k^{n})^{-1/m}. \tag{6.2}$$

$$z_{k+1} = \gamma a^{1/m+n}(1 - a^{-1} z_k^{m})^{1/n+m}. \tag{7}$$

In view of the similarity of these, rather than considering each one separately, write

$$z_{k+1} = \lambda(1 - \mu z_k^{r})^{1/s}, \tag{8}$$

and observe that, respectively,

$$\begin{array}{lll} \lambda = \alpha, & \beta a^{1/m}, & \gamma a^{1/n+m} \\ \mu = a, & 1, & a^{-1} \\ r = -m, & n, & m \\ s = n, & -m, & n+m. \end{array} \tag{9}$$

The process of iteration, then, applied to (8) gives successively

$$\begin{aligned} z_0 &= \lambda, \\ z_1 &= \lambda(1 - \mu\lambda^r)^{1/s}, \\ z_2 &= \lambda(1 - \mu\lambda^r(1 - \mu\lambda^r)^{r/s})^{1/s}, \\ z_3 &= \lambda(1 - \mu\lambda^r(1 - \mu\lambda^r(1 - \mu\lambda)^{r/s})^{r/s})^{1/s}, \text{ etc.} \end{aligned} \tag{10}$$

These iterations, or any power of them, can be expanded directly by the binomial theorem, assuming that $\mu\lambda^r$ is such that the expansions are legitimate.

Thus:

$$z_0^h = \lambda^h$$

$$z_1^h = \lambda^h \sum_p \frac{(-h/s)_p}{p!} (\mu\lambda^r)^p \ *$$

$$z_2^h = \lambda^h \sum_{p_1} \frac{(-h/s)_{p_1}}{p_1!} (\mu\lambda^r)^{p_1} (1 - \mu\lambda^r)^{(r/s)p_1}$$

*The customary notations:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad \sum_n A_n = \sum_{n=0}^{\infty} A_n$$

are adopted in this paper.

$$(11)\qquad =\lambda^h \sum_{p_1,p_2} \frac{(-h/s)_{p_1}\left(-(r/s)p_1\right)_{p_2}}{p_1!\ \ p_2!}(\mu\lambda^r)^{p_1+p_2}$$

$$z_3{}^h=\lambda^h \sum_{p_1,p_2} \frac{(-h/s)_{p_1}\left(-(r/s)p_1\right)_{p_2}}{p_1!\ \ p_2!}(\mu\lambda^r)^{p_1+p_2}(1-\mu\lambda^r)^{(r/s)p_2}$$

$$=\lambda^h \sum_{p_1,p_2,p_3} \frac{(-h/s)_{p_1}\left(-(r/s)p_1\right)_{p_2}\left(-(r/s)p_2\right)_{p_3}}{p_1!\ \ p_2!\ \ p_3!}(\mu\lambda^r)^{p_1+p_2+p_3}$$

And generally:

$$(12)\qquad z_k{}^h=\lambda^h \sum_{p_1,\cdots,p_k} \frac{(-h/s)_{p_1}\prod_1^{k-1}\left(-(r/s)p_i\right)_{p_{i+1}}}{\prod_1^k\ p_i!}(\mu\lambda^r)^{\sum_1^k p_i}.$$

This power of the kth iteration is clearly a power series in $\mu\lambda^r$, say:

$$(13)\qquad z_k{}^h=\lambda^h\sum_q A_q{}^k(\mu\lambda^r)^q,$$

where

$$q=\sum_1^k p_i \quad\text{and}\quad A_q{}^k$$

is the sum of the corresponding coefficients in (12). Since

$$(14)\qquad \frac{\left(-(r/s)p_i\right)_{p_{i+1}}}{p_{i+1}!}=\begin{cases}0, & p_i=0,\quad p_{i+1}\neq 0,\\ 1, & p_i=0\quad p_{i+1}=0,\end{cases}$$

it is possible to write, for $q\leqq k$

$$(15)\qquad A_q{}^k=\sum_{p_1=1}^{q}\ \sum_{p_2=1}^{q-p_1}\cdots\sum_{p_q=1}^{q-\sum_1^{q-1}p_i}\ \frac{(-h/s)_{p_1}\prod_1^{q-1}\left(-(r/s)p_i\right)_{p_{i+1}}}{\prod_1^q\ p_i!},$$

and for $q\geqq k$

$$(16)\qquad A_q{}^k=\sum_{p_1=1}^{q}\ \sum_{p_2=1}^{q-p_1}\cdots\sum_{p_{k-1}=1}^{q-\sum_1^{k-2}p_i}$$

$$\frac{(-h/s)_{p_1}\prod_1^{k-2}\left(-(r/s)p_i\right)_{p_{i+1}}}{\prod_1^{k-1}\ p_i!}\cdot\frac{\left(-(r/s)p_{k-1}\right)_{q-\sum_1^{k-1}p_i}}{(q-\sum_1^{k-1}p_i)!},$$

the summation over an index being replaced by one when the upper limit of summation is zero.

It is evident from (15) that A_q^k is independent of k for $q \leqq k$. Thus the form corresponding to (10) for iterations of order greater than k will differ only in the coefficients of powers greater than the kth. Furthermore, (15) may be written:

$$A_q^k = \sum_{p_1=1}^{q} \frac{(-h/s)_{p_1}}{p_1!} B_{p_1}^q \,, \tag{17}$$

where:

$$\begin{aligned} B_{p_1}^q &= \sum_{p_2=1}^{q-p_1} \sum_{p_3=1}^{q-p_1-p_2} \cdots \sum_{p_q=1}^{q-\sum_1^{q-1} p_i} \frac{\prod_1^{q-1} (-(r/s)p_i)_{p_{i+1}}}{\prod_1^{q-1} p_{i+1}!} \\ &= \sum_{p_2=1}^{q-p_1} \frac{(-(r/s)p_1)_{p_2}}{p_2!} B_{p_2}^{q-p_1} \,. \end{aligned} \tag{18}$$

Since $B_p^p = 1$, by (15) and (17), it follows by mathematical induction that

$$B_u^v = \frac{u}{v} \quad \frac{(-(r/s)v)_{v-u}}{(v-u)!} \qquad v \geqq u > 0 \tag{19}$$

for if (19) is assumed true for all $B_{p_2}^{q-p_1}$, $1 < p_2 < q - p_1$, a direct substitution of these values in (18):

$$\begin{aligned} B_{p_1}^q &= \sum_{p_2=1}^{q-p_1} \frac{(-(r/s)p_1)_{p_2}}{p_2!} \cdot \frac{p_2}{q-p_1} \cdot \frac{(-(r/s)(q-p_1))_{q-p_1-p_2}}{(q-p_1-p_2)!} \\ &= \sum_{w=0}^{q-p_1-1} \frac{1}{q-p_1} \cdot \frac{(-(r/s)p_1)_{q-p_1-w}}{(q-p_1-w-1)!} \cdot \frac{(-(r/s)(q-p_1))_w}{w\,!} \\ &= \frac{(-(r/s)p_1)_{q-p_1}}{(q-p_1)!} \sum_{w=0}^{q-p_1-1} \frac{(-(q-p_1-1))_w}{((r/s)p_1-q+p_1+1)_w} \cdot \frac{(-(r/s)(q-p_1))_w}{w\,!} \\ &= \frac{(-(r/s)p_1)_{q-p_1}}{(q-p_1)!} \cdot \frac{(-(r/s)q+1)_{q-p_1-1}}{(-(r/s)p_1+1)_{q-p_1-1}} \\ &= \frac{p_1}{q} \cdot \frac{(-(r/s)q)_{q-p_1}}{(q-p_1)!} = B_{p_1}^q \,, \end{aligned}$$

shows that (19) is then true for $B_{p_1}^{q}$. As in the usual argument the basic relation, $B_p{}^p = 1$, is then sufficient to proceed step by step to the value of $B_{p_1}^{q}$ for any p_1 and q.

Thus, for $0 < q \leq k$,

$$A_q{}^k = \sum_{p_1=1}^{q} \frac{(-(h/s))_{p_1}}{p_1\,!} \cdot \frac{p_1}{q} \cdot \frac{(-(r/s)q)_{q-p_1}}{(q-p_1)!}$$

$$= \frac{(-(r/s)q)_{q-1}}{q\,!} \sum_{p=0}^{q-1} \frac{(-h/s))_{p+1}}{p\,!} \cdot \frac{(1-q)_p}{(-q+(r/s)q+2)_p}$$

$$= \frac{h}{s} \cdot \frac{(-(r/s)q)_{q-1}}{q\,!} \sum_{p=0}^{q-1} \frac{(1-q)_p\,(1-(h/s)+1)_p}{(-q+(r/s)q+2)_p\,p\,!} -$$

(20)

$$= -\frac{h}{s} \cdot \frac{(-(r/s)q)_{q-1}}{q\,!} \cdot \frac{(-(h/s)-(r/s)q+1)_{q-1}}{(-(r/s)q)_{q-1}} \,{}^{*}$$

$$= -\frac{h}{s} \cdot \frac{(-(h+rq/s)+1)_{q-1}}{q\,!}$$

$$= \frac{h}{h+rq} \cdot \frac{(-(h+rq/s))_q}{q\,!} .$$

The hth power of the kth iteration, (13), may then be written:

$$(21) \quad z_k{}^h = \lambda^h \sum_{q=0}^{k} \frac{h}{h+rq} \cdot \frac{(-(h+rq/s))_q}{q\,!} (\mu\lambda^r)^q + R_k,$$

where

$$(22) \qquad R_k = \lambda^h \sum_{q=k+1}^{\infty} A_q{}^k (\mu\lambda^r)^q .$$

Write: $\quad \bar{z}_k{}^h = z_k{}^h - R_k$

$$(23) \qquad = \lambda^h \sum_{q=0}^{k} \frac{h}{h+rq} \cdot \frac{(-(h+rq/s))_q}{q\,!} (\mu\lambda^r)^q$$

Since z_k consists of that part of the iteration which is not affected by subsequent iterations, it will become and remain the dominant

*These summations have been carried out by use of the suitable form of the Gauss theorem:

$$\sum_n \frac{(-m)_n(\beta)_n}{(\gamma)_n n!} = \frac{(-m+\beta-\gamma+1)_m}{(-m-\gamma+1)_m}$$

part of the successive approximations to the power of the root of (3) if the iterative process is convergent. It can be further confirmed that the sequence $\bar{z}_k$, if convergent, will actually give the power of the root by writing, according to (23)

$$z^h = \lim_{k\to\infty} \bar{z}_k{}^h$$

$$= \lambda^h \sum_{q=2}^{\infty} \frac{h}{h+rq} \quad \frac{(-(h+rq/s))_q}{q\,!} \; (\mu\lambda^r)^q \tag{24}$$

whence

$$z^s = \lambda^s \sum_q \frac{s}{s+rq} \quad \frac{(-(s+rq/s))_q}{q\,!} \; (\mu\lambda^r)^q$$

$$= \lambda^s \sum_q \frac{1}{1+(r/s)q} \quad \frac{(-1-(r/s)q)_q}{q\,!} \; (\mu\lambda^r)^q$$

$$= \lambda^s - \lambda^s \sum_{q=1}^{\infty} \frac{(-(r/s)q)_{q-1}}{q\,!} \; (\mu\lambda^r)^q$$

$$= \lambda^s - \lambda^s \sum_q \frac{(-(r/s)(q+1))_q}{(q+1)!} \; (\mu\lambda^r)^{q+1}$$

$$= \lambda^s - \lambda^s \mu\lambda^r \sum_q \frac{1}{1+q} \quad \frac{(-(r+rq/s))_q}{q\,!} \; (\mu\lambda^r)^q$$

$$= \lambda^s - \lambda^s \mu z^r \tag{25}$$

$$= \lambda^s(1-\mu z^r)\,,$$

which by reference to (9) is equivalent to (3), so that:

$$z = \lambda \sum_{q=0}^{\infty} \frac{1}{1+rq} \quad \frac{(-1+rq/s))_q}{q\,!} \; (\mu\lambda^r)^q \tag{26}$$

is the root of (3).

Substituting the values given by (9) in (24), the explicit forms for the powers of the roots are shown to be respectively:

$$z^h = \alpha^h \sum_q \frac{h}{h-mq} \quad \frac{(-(h-mq/n))_q}{q\,!} \; (a\alpha^{-m})^q \tag{27.1}$$

$$z^h = \beta^h a^{h/m} \sum_q \frac{h}{h+nq} \quad \frac{((h+nq)/m)_q}{q\,!} \; (\beta^n a^{n/m})^q \tag{27.2}$$

$$(28) \quad z^h = \gamma^h a^{h/m+n} \sum_q \frac{h}{h+mq} \quad \frac{(-(h+mq/m+n))_q}{q\,!} \quad (\gamma^m a^{-n/n+m})^q.^*$$

These series will be recognized as identical with the series solutions obtained previously by the methods mentioned in the introduction.† Their properties have been fully discussed elsewhere.

Although questions of convergence have not needed consideration in the formal derivation of (27) and (28), the solutions are incomplete without explicit statements of their regions of validity. The ratio of two successive terms of the series (24) is given by

$$(29) \quad \frac{\dfrac{h}{h+r(q+1)} \cdot \dfrac{(-(h+r(q+1)/s))_{q+1}}{(q+1)!} \quad (\mu\lambda^r)^{q+1}}{\dfrac{h}{h+rq} \cdot \dfrac{(-(h+rq/s))_q}{q\,!} \quad (\mu\lambda^r)^q}$$

$$= \mu\lambda^r \frac{(1-(r/s))^{1-(r/s)}}{(-(r/s))^{-(r/s)}} \quad (1-(3/2q)+O(1/q^2))$$

according to the familiar properties of the Gamma function. The Weierstrass ratio test for series with complex terms‡ shows then that the series (24) is absolutely convergent when

$$\left| \mu\lambda^r \frac{(1-(r/s))^{1-(r/s)}}{(-(r/s))^{-(r/s)}} \right| \leq 1$$

or

$$(30) \quad |\mu\lambda^r| \leq \left| \frac{((r/s)-1)^{(r/s)-1}}{(r/s)^{r/s}} \right|$$

and divergent when

$$(31) \quad |\mu\lambda^r| > \left| \frac{((r/s)-1)^{(r/s)-1}}{(r/s)^{r/s}} \right|.$$

*Results identical with (27) and (28) are obtained as in the above discussion by considering the recurrence formulæ:

$$z_{k+1} = \alpha(1+az_k^{-(n+m)})^{-(1/n)}$$
$$z_{k+1} = \beta a^{1/m}(1+a^{-1}z_k^{n+m})^{1/m}$$
$$z_{k+1} = \gamma a^{1/m+n}\,(1-z_k^{-n})^{-(1/n+m)}$$

which are derived from (3) in the same manner as (6) and (7).

†NeKrassoff [2] p. 415, 418.

‡K. Knopp, Theorie und Anwendung der unendlichen Reihen, 2 aufl. theorem 229, p. 401.

Accordingly, by (9), when

$$|a| < \frac{m^{m/n} n}{(m+n)^{(m/n)+1}} \tag{32}$$

(27.1) and (27.2) converge absolutely and (28) diverges. When

$$|a| > \frac{m^{m/n} n}{(m+n)^{(m/n)+1}} \tag{33}$$

(27.1) and (27.2) diverge and (28) converges absolutely. On the circle of convergence

$$|a| = \frac{m^{m/n} n}{(m+n)^{(m/n)+1}} \tag{34}$$

(27) and (28) are all absolutely convergent.

The series (27) with appropriate values of α and β provide the $m+n$ roots in the region (32) while the series (28) with appropriate values of γ provide the $m+n$ roots in the region (33). It is well known that the series (27) and (28) are generalized hypergeometric functions and that (28) provide the analytic continuation of (27).* While both sets of series give the $m+n$ roots on the circle of convergence (34), the theory of hypergeometric functions* indicates that there are the singular points,

$$a = \frac{\alpha^m m^{m/n} n}{(m+n)^{(m/n)+1}} \tag{35}$$

for each function on this circle. At each singular point the equation clearly has the double root,

$$z = \frac{\alpha m^{1/n}}{(m+n)^{1/n}}, \dagger \tag{36}$$

by which information the equation can be reduced to a lower degree and reconsidered.

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*Appell, *Fonctions Hypergeometriques et Hyperspheriques.*
†See Dobrzycki [1] for this same result in the case of real roots.

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[6] *Auflösung der allgemeinen Gleichung* n^{ten} *Grades durch convergent Reihen,* (Abhandl. und Berichte der Techn. Staatslehranstalten in Chemnitz, Heft 5, pp. 27-46, 1915).

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A Generalization of Euler's Relation in the Triangle

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1. Let P and Q be two isogonal conjugate points as to the triangle $A_1A_2A_3$ inscribed in a circle having as center O and as radius R.

If, in complex coordinates with origin O, x and y are the coordinates of P and Q and $\bar{x}$ and $\bar{y}$ the conjugates of x and y, the radius of the common pedal circle of P and Q as to the triangle $A_1A_2A_3$ is given by*

$$4R^2\rho^2=(R^2-x\bar{y})(R^2-\bar{x}y).$$

The relation exists whatever the position of the base line may be as to the circle Γ. We will now choose that line such that the point Ω having as coordinate R lies on the internal bisectrix of the angle POQ, which will be denoted by θ.

In that case, we find

$$\bar{y}=(OQ/OP)x, \quad y=(OQ/OP)\bar{x}$$

and
$$4R^2\rho^2=\left(R^2-\frac{OQ}{OP}x^2\right)\left(R^2-\frac{OQ}{OP}\bar{x}^2\right)$$

$$=\left(R-x\sqrt{\frac{OQ}{OP}}\right)\left(R-\bar{x}\sqrt{\frac{OQ}{OP}}\right)\left(R+x\sqrt{\frac{OQ}{OP}}\right)\left(R+\bar{x}\sqrt{\frac{OQ}{OP}}\right);$$

the last expression is the product of the squares of the distances from Ω to the points taken on OP at distances $\pm(OP\cdot OQ)^{\frac{1}{2}}$ from O. Hence

$$4R^2\rho^2=[OP\cdot OQ+R^2-2R(OP\cdot OQ)^{\frac{1}{2}}\cos\theta/2]$$
$$x[OP\cdot OQ+R^2+2R(OP\cdot OQ)^{\frac{1}{2}}\cos\theta/2],$$

or
$$4R^2\rho^2=R^4+2R^2OP\cdot OQ+\overline{OP}^2\cdot\overline{OQ}^2-4R^2OP\cdot OQ\cdot\cos^2\theta/2$$
$$=R^4+\overline{OP}^2\cdot\overline{OQ}^2-2R^2OP\cdot OQ\cos\theta.$$

*F. and F. V. Morley, *Inversive Geometry*, p. 197.

If P and Q are two isogonal conjugate points as to a triangle inscribed in a circle having as center O and as radius R, the radius ρ of the common pedal circle of P and Q as to the triangle is given by

$$4R^2\rho^2 = R^4 + \overline{OP}^2 \cdot \overline{OQ}^2 - 2R^2 OP \cdot OQ \cos\theta.$$

θ being the angle POQ.

We have also

$$4R^2\rho^2 = R^4 + \overline{OP}^2 \cdot \overline{OQ}^2 - R^2(\overline{OP}^2 + \overline{OQ}^2 - \overline{PQ}^2). \tag{1}$$

2. When P is the center I of one of the circles tangent to the sides, $\cos\theta = 1$ and

$$\pm 2R\rho = R^2 - \overline{OI}^2.$$

When P and Q are the circumcenter and the orthocenter $\rho = R/2$.

When P and Q are the Brocard points, then, if ω is the Brocard angle

$$4\rho^2 = R^2[1 + (1-4\sin^2\omega)^2 - 2(1-4\sin^2\omega)(1-2\sin^2\omega),$$

or $\rho = R \sin\omega$.

3. It is well known that P and Q are the foci of a conic inscribed to the triangle having 2ρ as major axis. The relation* shows that when P and Q are supposed to be given as also R and ρ, the locus of O is such that

$$\overline{OP}^2 \cdot \overline{OQ}^2 - R^2(\overline{OP}^2 + \overline{OQ}^2)$$

is constant; hence the locus of O is then a leminscate of Booth,† pedal curve of a conic as to its center:

The locus of the circumcenter of the triangles circumscribed to a given conic and having a circumcircle of a given radius is the pedal curve of a conic with respect to its center.

*R. A. Johnson, *Modern Geometry*, p. 270.

†G. Loria, *Spesielle ebene Kurven*, Vol. I, p. 135.

On Roots of Unity

By Grace Shover
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In this paper we prove three simple theorems in the theory of rational integers and then apply the results to proving theorems on the product of the n nth roots of unity, the product of the primitive roots of unity, and the product of the imprimitive roots of unity.

1. *Some theorems on integers.*

Theorem 1.1. The sum of the first n integers is a multiple of n if and only if n is odd. Twice this sum is always a multiple of n.

Proof. The sum of the first n integers, $n(n+1)/2$, is a multiple of n if and only if n is odd, but obviously twice this sum is always a multiple of n.

Crelle* seems to have been the first to note the

Lemma. The sum of all the integers less than n and prime to n is $n\,\Phi(n)/2$ where $\Phi(n)$ is the Euler Φ-function.

Proof. The integer $a<n$ is prime to n if and only if $n-a$ is also. Thus with every integer less than n and prime to n there is paired another integer less than n and prime to n such that their sum is n. There are $\Phi(n)$ such integers in all. Therefore, their sum is $n\,\Phi(n)/2$.

Theorem 1.2. The sum of the integers less than or equal to n and prime to n is a multiple of n if $n>2$.

Proof. If $n=p_1^{a_1}\cdots p_k^{a_k}$ where the p_i are distinct primes, then

$$\Phi(n)=n\frac{p_1-1}{p_1}\cdots\frac{p_k-1}{p_k}=p_1^{a_1-1}\cdots p_k^{a_k-1}(p_2-1)\cdots(p_k-1).$$

Thus $\Phi(n)$ is even if $n\neq 2^h$ for at least one factor p_i-1 is even. If $n=2^h$, then $\Phi(n)=2^{h-1}$ which is also even if $h>1$. Obviously, the theorem fails if $h=1$, that is, if $n=2$.

Theorem 1.3. The sum of the integers less than or equal to $n>2$ and not prime to n is a multiple of n if and only if n is odd. Twice this sum is always a multiple of n.

Proof. Let S_i be the sum described in Theorem 1.1. If n is odd, then $S_1=kn$ and $S_2=ln$ where k and l are integers. Therefore,

*Encyklopädie der Zahlentheorie, Jour. für Math., 29, 1845, pp. 58-95.

$S_3=(k-l)n$. If n is even, then $2S_1=kn$ and $S_2=ln$ where k is necessarily odd. Then $S_3=(k-2l)n/2$, half an odd multiple of n. However, $2S_3$ is always a multiple of n.

2. *Roots of Unity.* The n nth roots of unity are expressible in the form

$$R^k=\cos\frac{2\pi k}{n}+i\sin\frac{2\pi k}{n}\cdot \qquad (k=1,\cdots n)$$

The primitive nth roots of unity are those such that no power of the root is 1 where the exponent is not a multiple of n. Hence the primitive nth roots of unity are of the form R^k where the greatest common divisor of k and n is 1.

Theorem 2.1. The product of the n nth roots of unity is 1 if n is odd and -1 if n is even.

The proof follows from Theorem 1.1.

Theorem 2.2. The product of the primitive nth roots of unity is 1 if $n>2$.

The proof follows from Theorem 1.2.

Theorem 2.3. The product of the imprimitive nth roots of unity is 1 if n is odd and -1 if n is even except for the case $n=2$.

"In my correspondence with authors in regard to revision of papers, I have stressed the mportance of condensation without omission, but it seems that the ability to express oneself tersely is one which many writers lack. Our publication is not the only one publishing papers with the fault of wordiness. Most mathematical papers can be cut to almost half their length without loss of anything essential, and with a distinct gain in clarity. A great gain can be made by the deletion of repetitions, unnecessary illustrations, and numerous special cases leading up to general theorems which in themselves include al' that has gone before. All of these practices may be good pedagogy, but isn't it true a pedagogical approach in a scientific paper is likely to strike the reader with an already trained imagination as a waste of his time. I can suggest no remedy for this fault. My own efforts to get the point over to authors have resulted in nothing more than having them make the specific changes which I suggest without at all improving the general tone of the paper."—By L. E. Bush, in an invited criticism of this journal.

A Third Lesson in the History of Mathematics

By G. A. Miller
University of Illinois

7. *Quadratic equations.* Forward steps toward the modern solutions of the general quadratic equation are now known to have been taken at various times during a period of about four thousand years, beginning with work by the ancient Sumerians and the ancient Babylonians and extending to recent times. These steps were taken in many different countries and exhibit the fact that our subject was brought to its present stage of development by the cooperation of very many different people living at widely separated periods of time and in numerous countries. One may be surprised at first when one learns that a partial solution of the quadratic equation was considered so early in the intellectual history of the human race since it involves the consideration of inverse operations and encounters difficulties in regard to the extensions of the number concept so as to include the negative and the complex numbers.

These have a long history and have a fundamental bearing on many other questions of the history of mathematics. Hence they may properly be included among the primary facts of this history since they are very rich in their implications*. When the modern student solves the general quadratic equation by completing the square, for instance, he is using a method which was used by the ancients about four thousand years ago. The difference is that he is not now halted by the appearance of a negative number under the radical sign as the ancients were. This stumbling block for the ancients became one of the most fruitful sources of progress in mathematics for centuries from the time when the Italian mathematicans, including H. Cardan (1501-1576), started to remove it. Hence one may well be astonished when one meets

*G. A. Miller, *Collected Works*, vol. 2, pp 493-526, 1938.

such statements as the following, "so that the general quadratic as we know it today was thus fully mastered by the Greek mathematicans."*

Even the consideration of the negative roots of quadratic equations is posterior to the Greek period of mathematical developments. In fact, these Greeks never gave more than one root of the many quadratic equations considered by them, but it seems possible that some of them realized that certain quadratic equations have as many as two positive roots. The greatest merit of the work of the Greeks as regards the solution of the general quadratic equation is that they showed how to construct geometrically a root of such an equation when this root is positive. No instance is known where they considered the solution of a quadratic equation when the two roots of such an equation are negative. This throws additional light on the quotation noted near the close of the preceding paragraph and emphasizes the fact that some of the mathematics now taught in our secondary schools is really quite modern. The first known instance of the solution of a quadratic equation having its two roots negative appeared in A. Girard's *Invention nouvelle* (1629).

Many modern high school students realize that it is very easy to prove that a quadratic equation cannot have more than two distinct roots but no proof of this fact appears in the mathematical literature before the time of the English mathematician T. Harriot (1560-1621), although the German mathematician M. Stifel stated earlier in his famous *Arithmetica integra* (1544) that a quadratic equation cannot have more than *two distinct* roots. It should be noted that a proof of this fundamental theorem is based on the fact that the solution of the quadratic equation is equivalent to the factoring of a quadratic expression but this equivalence does not seem to have been recognized before the sixteenth century, that is, more than three thousand years after the anceint Sumerians and the ancient Babylonians used general methods to solve partially the quadratic equations but were unable to find a correct interpretation of some of the results involved in their work. The generalization of methods which proved successful in special cases has been one of the most fruitful sources of mathematical progress.

In 1934 a noted German writer on the history of elementary mathematics, J. Tropfke, considered various forward steps in the solution of the quadratic equation during a period of about thirty-five hundred years without including the modern methods to find its complex roots graphically. He published then a long article on this subject in the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, which was largely embodied in the third edition of volume 3 (1937) of his well

*D. E. Smith, *History of Mathematics*, vol. 1, p. 126, 1923.

known *Geschichte der Elementar-Mathematik*, in which 68 pages are devoted to this subject. In the second edition of this volume, which appeared about 15 years earlier, only 27 pages were devoted thereto. This is evidence of the fact that our knowledge of the history of the quadratic equation has been greatly extended during recent years. These extensions are mainly due to a better understanding of the algebraic work of the ancient Sumerians and the ancient Babylonians, and they exhibit the fact that much of our knowledge relating to secondary mathematics is comparatively recent and does not appear in our common histories of mathematics.

While the history of the quadratic equation extends through a long period of time the known advances sometimes occurred at long intervals. In particular, for about 1500 years at the beginning of this long period of time no advances are known to have been made until about the time of Euclid. After the early Greek contributions there again appears a considerable period of inactivity as regards important advances until the Hindus studied all the quadratic equations under a single type instead of under various different general types as the ancient Greeks had done. Many of the followers of the latter continued to do so for about a thousand years until the time of M. Stifel, even after the ancient Hindus had exhibited the better way. In the sixteenth and seventeenth centuries progress was relatively rapid but the early part of the nineteenth century had to be awaited for a common knowledge of a clear treatment of the complex numbers, without which the treatment of the quadratic equation was necessarily incomplete.

Even the algebraically partial solution of the quadratic equation implies a somewhat advanced stage of civilization, and the fact that the ancient Babylonians had reached this stage is of fundamental importance in the history of mathematics. No evidence has yet been produced which tends to prove that the aborigines of any country on the American continent reached such an advanced stage of mathematical development before they came in contact with Europeans. The fact that some of the methods used by the ancient Babylonians are still in use and can be employed to find both of the roots of a quadratic equation deserves emphasis since mathematical progress has frequently been inspired by the unforeseen fertility of the methods used in special cases. While the ancient Greeks failed to give reference to the work of the ancient Babylonians in their extant writings relating to the quadratic equation it seems likely that these writings were greatly influenced by this older work.

To the American student of mathematics it may be especially interesting that the quadratic equation received some attention in a work

printed in America during the sixteenth century, viz., the *Sumario* by Juan Diez Freyle, 1556. No negative roots were given therein for the solutions of the equations. In fact, all the problems were very elementary and did not involve the consideration of imaginary roots. Near the end of this little volume the author remarked that he wished merely to set down the things which were necessary and familiar. The algebraic part of the volume covers only about six pages and is much less advanced than the *Ars Magna* of H. Cardan which was published in Europe about eleven years earlier. Perhaps the most important fact for the student of the history of the quadratic equation is to bear in mind that formal developments of our subject frequently contained much which was only partially understood for a long time and became a great incentive for later work. Even in the ancient Babylonian mathematics this incentive is apparent.

8. *Approximations regarded as exact.** One of the primary facts of the history of mathematics is that approximate results were frequently regarded for long periods of time as accurate. A striking example of this is the use of the number 3 as the value of the ratio of the circumference of a circle to its diameter. This value was extensively employed and it is an illustration of a tendency even in mathematics to accept statements as true which can easily be disproved. In constructing wheels for various purposes in ancient and medieval times it seems reasonable that some of the workmen observed that the given ratio exceeds 3 but their influence on the intellectual world was probably less than it is today, and they were less likely to be familiar with the fact that the number 3 was widely accepted as the value of this ratio than are the workmen of the present day. Hence the persistence of the number 3 for this ratio throws light on the disparity of intellectual influences in former times.

The fact that none of the ancient and medieval workmen are now known to have reported that the ratio of the circumference of a circle to its diameter exceeds three, although closer approximations were also in use, appears less surprising than that the number 3 as a value of this ratio has not been expunged from all of the widely used sacred writings up to the present time. The wise admonition "Prove all things: hold fast that which is good" was not always completely followed by the writers of our Bible even as regards mathematical questions, as may be seen from the following quotation: "And he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits in height thereof; and a line of thirty cubits did compass it

*This section is based on an article by the present writer published in the *Mathematics Student* (India), vol. 6, 1938.

round about." If the distance around had been actually measured it would have been found to be almost thirty-one and a half cubits instead of thirty cubits as stated in this quotation.

A large number of other approximate values for the ratio of the circumference of a circle to its diameter were employed by the ancient and medieval mathematicians. One of these is deduced from the fact that the ancient Egyptians assumed that the area of a circle is equal to 8/9th of the diameter of the circle. This is equivalent to assuming that the value of the given ratio is 256/81 which is a much closer approximation than the number 3 and has been used as evidence tending to show that the ancient Egyptians were better mathematicians than the ancient Babylonians, since the latter commonly employed the number 3 as the value of this ratio. Many other evidences, however, show that the ancient Babylonians were superior to the ancient Egyptians along mathematical lines. The outstanding mathematical achievement of the latter is that they had a correct rule to find the volume of the frustum of a square pyramid but they were probably unable to prove the accuracy of this rule, judging from their other mathematical attainments.

An approximate value of the ratio of the circumference of a circle to its diameter which is closer than the ancient Egyptian value noted above, and seems to have largely superseded this value in ancient and medieval times, was given by Archimedes (287-212 B. C.) in his *Measurement of the Circle*. It was at first explicitly given as an approximation but was later widely assumed to be accurate and is still widely used when only rough approximations are required. Its simple form 22/7 makes its use very convenient and it was rapidly adopted in other countries, being used in China as early as the fifth century of our era. The fact that this ratio is a transcendental number naturally prolonged the period during which various approximations to its true value were assumed to be correct, since the ancient and the medieval mathematicians knew nothing about the existence of such numbers and hence were unable to explain the difficulties involved in this particular ratio. The fact that it is transcendental was first established by F. Lindemann in 1882, and it is definite evidence of the superiority of modern mathematics.

The use of the number 3 for the ratio of the circumference of a circle to its diameter appears also in the Koran and it is encouraging that no one is known to have been persecuted on account of his efforts to find closer approximations to the true value of this ratio. The comparatively little interest in mathematical questions on the part of the general public in ancient and medieval times is clearly exhibited by

the persistence of the use of the number 3 as this ratio in commonly used sacred writings as well as by the comparatively small space devoted to this subject in the writings which have come down to us from ancient times. Only about 250 of the several hundred thousand cuneiform texts which are now in the various museums of the world are known to be devoted to mathematical tables or to actual mathematical questions, according to O. Neugebauer's *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften*, volume 1, 1934, page 206, which is largely based on source material.

Many statements which are only approximations but are not expressed as such still appear in the literature of logarithms. For instance, in such a widely used work of reference as the Weber-Wellstein *Enzyklopädie der Elementarmathematik* it is stated* that Napier's logarithm of a number is its logarithm to the base $1/e$. Important facts in this connection are that while logarithms to the base $1/e$ do not differ in the first few decimals from some of those found in the tables by Napier they are essentially different since the latter were not computed with respect to any base. The statement is also very misleading since the number e was not determined until after the days of Napier. Various writers have assumed that the base of a table of logarithms could be determined by means of the numbers given for the logarithms which appear in this table. This is obviously impossible since an infinite number of bases which differ only slightly would give rise to the same logarithms as those found in such a table to a given number of decimals.

Approximations to the roots of given numbers, especially to square roots, were naturally common among the ancients, since the roots of rational numbers are usually irrational and the existence of irrational quantities was first established by the ancient Greeks who, however, did not commonly assume the existence of irrational numbers. A cuneiform Babylonian text contains a rule which is the equivalent of the formula $\sqrt{a^2+b}=a+b/2a$, and rules equivalent to this approximation formula were very extensively used in the ancient mathematical literature, including that of the Greeks. In many cases a and b can be so selected that this formula gives a sufficiently close approximation for certain practical purposes. In the special case of the square root of 2 an unusually close approximation appears in the ancient Hindu literature in the following form:

$$1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}$$

*Vol. I, 4th edition, p. 148 (1922).

This gives the first five decimals correctly and is one of the most remarkable approximations which is known to appear in the ancient Hindu literature. It has given rise to extensive speculation as regards its possible origin.

One great difficulty in regard to approximate results in the history of mathematics is that it is frequently impossible to determine definitely whether certain results given by an author are supposed to be accurate or only approximate. A large part of mathematics is devoted to approximations which can often be carried so far that the difference between the true value and the one actually given could not be determined by the available instruments of measure. In elementary mathematics the subject of trigonometry is largely devoted to the study of proximate results. The student is thus led early to the use of approximations and it is often considered unnecessary to state explicitly that the results obtained are only approximately true. The mathematician is thus led to leave to the reader in many cases the decision whether certain results given by him are to be regarded as accurate or only approximate and various critical points are naturally obscured thereby, since many readers do not assume responsibilities involving thinking until they are asked to do so.

Even in such simple questions as the proof of the existence of irrational quantities the mathematician insists on a degree of exactitude which can never be reached by measurement, while in others, such as the use of tables of logarithms, he is satisfied with approximations which are frequently not very close. *This dual character of mathematics naturally complicates its history.* The ancient Greeks seem to have been the first to recognize this dual nature of mathematics and to have largely but not entirely confined their attention to the exact side of mathematics, especially after their discovery of irrational quantities. In applied mathematics, on the contrary, it is commonly necessary to deal with approximations and frequently the data to which mathematical reasonings are applied are in themselves such rough approximations as to make exact methods impracticable. Most Greek writers failed to consider such applications. In particular, they were not considered in Euclid's *Elements.*

Since our Bible contains very little relating to mathematics it may be interesting to note that what may be regarded as the most astounding error in the history of mathematics, viz., the long continued and widely extended use of 3 as the ratio of the circumference of a circle to its diameter, is found therein and is nowhere corrected therein. The mathematical historian is especially interested in such facts since they help him to form a satisfactory view as regards the

disparity of mathematical knowledge, and the difficulty of correcting errors after they have become widely accepted as true. The numerous contradictions in the ancient literature are partly responsible for the critical attitude of the modern historian which is now commonly regarded as essential for real progress in the history of mathematics.

9. *Fundamental laws of combination.* The associative law, the commutative law, and the distributative law are fundamental in the combination of the symbols of elementary algebra and are frequently emphasized in the freshmen courses therein. Each of these laws received its now most common name during the first 60 years of the nineteenth century. The last two were thus named by F. Servois in volume 5 page 98 (1814) of one of the earliest good mathematical periodicals, entitled *Annales de mathématiques,* which appeared in France in 22 volumes (1810-1831) and was a model for various later mathematical journals in view of its relatively high standards. The associative law of addition and multiplication was frequently mentioned in the *Lectures on Quaternions* (1853) by W. R. Hamilton, the most noted among the Irish mathematicians up to the present time, whose *Collected Works* began to appear in 1931 under the auspices of the Royal Irish Academy. In the preface to these "Lectures" its author stated, on page 61, that "to this associative principle or property of multiplication I attach much importance."

The associative law applies only to the case when three or more quantities are combined. In the case of three quantities $a, b, c,$ it may be expressed symbolically with respect to addition as follows:

$$a+(b+c)=(a+b)+c$$

With respect to multiplication it is commonly expressed symbolically as follows:

$$a(bc)=(ab)c$$

If it applies to every three quantities of a set of three or more quantities it can easily be proved that it applies to an arbitrary number of these quantities. This law, like many other laws in mathematics was implicitly used for a long time before it received a special name. In fact, it was explicitly noted by a well known French mathematician A. M. Legendre (1752-1835) in the third edition of his *Théorie des nombres,* page 2 (1830). The first edition of this work appeared in 1798 and was the earliest extensive treatise on the subject. It was later greatly enlarged and published in two volumes.

This law is especially useful in abstract mathematics when the elements which are combined are not restricted except that they are

supposed to obey certain laws of combination. In fact, in the theory of groups it has frequently been assumed that the elements must obey the associative law, but not necessarily the commutative law, when they are combined. The associative law thus became on the part of various writers an essential law of the vast subject of group theory. Not all writers on the subject have always adhered to this limitation, but no extensive abstract group theory has thus far been developed in which this law was not implied in some of the developments.

In the combination of our ordinary numbers each of the given three laws is always satisfied. The first extensive subject to be developed in which the commutative law is not always satisfied is the theory of substitution groups, or permutation groups, which was developed by J. L. Lagrange (1736-1813), A. L. Cauchy (1789–1857) and many others in view of its usefulness in the theory of algebraic equations. In this theory the associative law is always satisfied and the development of this theory contributed much towards exhibiting the fundamental importance of this law and towards making it later a central feature in the theory of abstract groups, which "investigates an algebraic operation in its purest form" according to the Russian mathematician L. Pontrjagin, *Topological Groups*, page 3 (1939), and was partly inaugurated by the English mathematician A. Cayley (1821-1895).

According to the associative law we may combine an arbitrary element of a set of three or more elements with one of its adjoining elements in a given order without affecting the rest of the elements of the set. The order in which the two elements in question are combined is important since the commutative law, which applies also to the combination of two elements, is not necessarily satisfied when the associative law is assumed to be satisfied. Neither is the associative law necessarily satisfied if the commutative law is assumed to be satisfied as will be seen later in this article. There is, however, not yet any large and useful subject in which the commutative law but not the associaative law is assumed to be satisfied, while the reverse is true when the associative law but not the commutative law is satisfied, as results from the theory of substitution groups noted above.

It should be noted that the associative law does not relate to the manner in which two elements of the set are combined but only to the manner in which the elements are associated when the number of the elements of the set is at least three. Hence the associative law of the addition of ordinary numbers is the same as the associative law of the multiplication of these numbers. This is contrary to the view expressed in the *Encyclopaedia Britannica* (1938) under the entry "As-

sociative laws", where it is stated that they are "two laws relating to numbers, one with respect to addition and the other with respect to multiplication". A similar remark appears under the entry "Commutative laws", in the same encyclopedia. These remarks are contradicted implicitly later in the same work under the entry "Quaternions", volume 18, page 835. In view of the fact this encyclopedia is very widely used and has been widely advertised as accurate it seems remarkable that such obvious and fundamental errors, which are not due to oversights, appear therein.

While the associative law applies only to the combination of three or more quantities of a set the commutative law relates also to the combination of two of these quantities and hence it prescribes the order in which these quantities are to be combined. It may be illustrated by the movements of a plane equilateral triangle which transform this triangle into itself. Such a triangle has three lines of symmetry about which we may rotate the triangle through 180° so as to transform it into itself. If we thus rotate successively around two of three distinct lines the result will be equal to the rotation of the triangle in its plane through 120°, but the latter rotation depends upon the order in which the former rotations are performed, and hence the former rotations are not commutative.

This constitutes one of the simplest illustrations of two non-commutative operators, which can be comprehended by every one. It is well known that the equilateral triangle is the first figure treated in Euclid's *Elements* but the non-commutativity of the movements which transform it into itself is not noted therein. It is however, noted in Book vii of these *Elements* that $ab=ba$ in the multiplication of abstract numbers, which may be regarded as the most fundamental theorem in regard to the combination of two numbers by multiplication and shows that the commutative law is satisfied in this multiplication. The fact that such a common law as the commutative law did not receive a special name in the mathematical literature before the early part of the nineteenth century exhibits the slowness with which mathematics developed along general lines even with regard to some of the most elementary observations of its principles.

As a simple illustration of the fact that a set of elements may obey the commutative law when they are combined without obeying also the associative law we may note the following somewhat artificial method of combining the symbols 1, 2, 3:

$$\begin{array}{lll} 1\cdot 1=1 & 1\cdot 2=3 & 1\cdot 3=2 \\ 2\cdot 1=3 & 2\cdot 2=2 & 2\cdot 3=1 \\ 3\cdot 1=2 & 3\cdot 2=1 & 3\cdot 3=3 \end{array}$$

It is obvious that $(1\cdot 2)3 \neq 1(2\cdot 3)$ according to these laws of combination. While this example is artificial it may serve to illustrate the great importance of assuming the associative law in the combination of elements in abstract mathematics and to show that in combining elements in mathematics which have definite properties these properties may make it unnecessary to define some of the laws of combination and hence these laws have often not been explicitly announced.

The distributive law is symbolically expressed as follows:

$$a(b+c+d+\cdots)=ab+ac+ad+\cdots.$$

It therefore relates to both of the two operations of addition and multiplication in an explicit form. It is clear from what precedes that these operations need not be restricted to the combination of numbers. In the combination of numbers the two methods known as addition and multiplication arose very early and one of these operations was commonly denoted by simple juxtaposition. From the earliest times, when only the symbol for unity seems to have been used, the juxtaposition of numerical symbols implied addition and in our modern Roman numerical notation juxtaposition still commonly implies addition, but the symbols can as a rule not be changed either according to the associative law or according to the commutative law without affecting the meaning of the symbols.

The ancient Babylonians and the ancient Egyptians implied addition by the juxtaposition of their numerical symbols while we now more commonly imply multiplication thereby. For instance, ab is now commonly assumed to mean $a\cdot b$ but not $a+b$. The Hindus already represented multiplication by juxtaposition but it is not likely that this practice influenced the adoption of this method later in Europe. When coefficients were used by Vieta (1540-1603) and others it was natural to represent multiplication by juxtaposition without any symbol to denote the operation. The ancient Greeks, like their predecessors, usually implied addition when they placed two number symbols together without any symbols of operation between them and we also do this sometimes implicitly with respect to our common numerical positional arithmetic, since 27 implies $20+7$. In a more explicit form it is involved when $12\frac{1}{2}$ is regarded as $12+\frac{1}{2}$.

A History of American Mathematical Journals

By Benjamin F. Finkel
Drury College

(Continued from January, 1941, issue)

THE
CAMBRIDGE MISCELLANY
of
MATHEMATICS, PHYSICS, AND ASTRONOMY.
1842
THE CAMBRIDGE MISCELLANY OF MATHEMATICS,
PHYSICS, AND ASTRONOMY
1842

THE CAMBRIGDE MISCELLANY OF MATHEMATICS,
PHYSICS, AND ASTRONOMY
To be continued quarterly
APRIL, 1842

Edited by Benjamin Peirce, A. M., Professor of Mathematics and Natural Philosophy in Harvard University.

No. I

Boston: Published by James Monroe and Company: John Owen, New York and London; Wiley and Putman, Philadelphia: Thomas Coperthatie and Co., Baltimore: Cushing and Bros., Charleston; Samuel Hart, Senior. MDCCCXII.

Cambridge Press:—Thurston and Torry

The Cambridge Miscellany of Mathematics, Physics, and Astronomy is divided into two departments; A Junior Department of Mathematics and a Senior Department of Mathematics.

Chapter I. *Junior Department of Mathematics.* Five problems are taken from the *Mathematical Miscellany* and five others, are proposed for solution. Rules of False or Double Position.

Chapter II. Six solutions are from the *Mathematical Miscellany* and Six problems are proposed for solution. Note on Sturm's Solution of Numerical Equations.

Chapter III. *Astronomy and Physics.* American Astronomical and Magnetic Observations, by the Editor. Distances of the Fixed Stars, by the Editor.

Chapter IV. *Meteors and Meteorology.* Meteors, by the Editor. Variety of climate. The Barometer, by the Editor.

THE CAMBRIDGE MISCELLANY OF MATHEMATICS,
PHYSICS, AND ASTRONOMY
To be continued quarterly
JULY, 1842

Edited by Benjamin Peirce, A. A. S., Perkins, Professor of Astronomy and Mathematics in Harvard University.

Boston:—Published by James Monroe and Co.

No. II. Contains a Junior and a Senior Department in Mathematics. Chapter V is devoted to the solution of problems proposed in the Junior Department in No. I.

Chapter VI. Senior Department in Mathematics contains the solutions of the six problems proposed in the previous number and six other problems proposed for solution.

Chapter VII. *Astronomy and Physics.* In this chapter is an article entitled, "On the Applications of Mathematics to Researches in the Physical Sciences," by Professor Lovering; and an article, "Observations on the Diurnal Vibration of the Magnetic Needle," by Dutrochet, translated from *Comptes Rendus*, etc., 1841; Encke's Comet, by Professor Lovering and extract from a letter of Professor Encke's of Berlin, dated December 20th, 1841.

No. III., bears the date, October 1842. Chapter IX., Junior Department in Mathematics, contains the solutions of the five problems proposed in No. II., and five new problems proposed for solution.

Chapter X. Senior Department in Mathematics contains the solutions of the problems proposed in No. II and six new problems proposed for solution.

Chapter XI. *Astronomy and Physics.* Contains a continuation of Professor Lovering's article, "On the Applications of Mathematics to the Physical Sciences," and a "History of the Present Magnetic Crusade," by Humphrey Lloyd, Professor of Natural Philosophy in the University of Berlin.

Chapter XII. *Meteorology.* Contains an article "On the Laws of Atoms", by H. W. Dove, of Berlin.

No. IV. Bears date, January, 1843.

Chapter XIII. Junior Department of Mathematics. Contains solutions of the five problems proposed for solution in No. III and five new ones are proposed for solution.

Chapter XIV. Senior Department in Mathematics. Contains solutions of the six problems proposed in No. III and six new problems proposed for solution.

Chapter XV. *Astronomy and Physics.* Contains an article entitled, "The Divisibility of Matter", by Professor Lovering; Memoir on "The General Principle of Natural Philosophy", by M. G. Lamé, Professor at the Polytechnic School, from the *Comptes Rendus.*

January 3, 1943. Chapter XVI. *Meteorology.* An article on "Atmospheric Electricity". Read by Mr. Spencer at the Polytechnic Institute, Liverpool.

The whole number of pages in the *Cambridge Miscellany of Mathematics, Physics, and Astronomy* is 192.

Our further information on this journal is no more enlightening than was that given John Stuart Mill when informed by the proposition that Humpty Dumpty was an Abracadabra.

"When a boy, I was able with a little help of a teacher who had never heard of a Teachers' College, to survey large plots of ground in different ways and derive the same acreage; from observation made below we were able to compute the height of a neighboring mountain, and we found the width of a river without crossing it. In these measurements we were interested in comparing the results made by multiplication, logarithms, and the slide rule. My first lesson in trigonometry was given to me when five years old by my mother's colored carriage-driver, who could neither read nor write. The problem was to find how high a cat was in the crotch of a big mulberry tree which I could not climb. We cut a stick which was exactly my height, no matter what that was. I was then instructed to get back lying on the ground until I could see the cat over the end of the stick which I held between my heels and perpendicular to the ground. The distance from my head to the tree gave the height of the cat in the tree."—From Harris Hancock's *Prerequisite Requirement for a College of Liberal Arts* (Part I).

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Coordinating the Teaching of Mathematics in High Schools and Colleges*

By DWIGHT F. GUNDER
Colorado State College, A & M

In a discussion of so important a problem as the coordination of the teaching of mathematics at the several levels in our educational system it would seem well to look rather critically at the two basic phases of the problem. These two phases are the subject matter presented and the methods of presentation. If the essence of the problem is found to be almost entirely in one of these two phases a focal attack may then be directed upon it.

The first step in this study consisted in finding what discrepancies might exist between the material presented in the high school courses in algebra and geometry (or their equivalents) and the material expected of entering freshmen by the various colleges and universities of this region. As was in part suspected from the start it was found that there was little if any such discrepancy. The material presented in the high school courses is for the most part exactly what is desired by the college teachers. A few minor differences may exist but certainly these are not of sufficient importance to cause the present unhappy and unhealthful condition.

The problem resolves itself then into the fact that the chief fault lies not in the nature of the material presented but rather in the failure on the part of the students to retain this material. That this is the crux of the matter is not surprising to any one who has taught a subject with extreme care only to discover a year, a semester or even a month later that the student recalls almost nothing of that which we were so sure he knew such a short time before. What then is the explanation

*A paper based upon the report of the "Committee for the Coordination of the Teaching of Mathematics in the High Schools and Colleges of Colorado and Wyoming." Members: H. W. Charlesworth, C. A. Hutchinson, Edward Morey, O. H. Rechard, Ethelyne B. Rhiner, Dwight F. Gunder, (Chairman).

of the fact that when the student is in high school he can solve equations up to and including simple quadratics while one or two years later he is unable to solve the simplest linear equation? Certainly something can be done to increase retention to a higher level than this. A few remedial measures are suggested in the following paragraphs. Any good teacher can add many more.

The first suggestion is based upon the hypothesis (one which is fairly well accepted) that a student will retain foı a longer period that which he has understood than that which he has acquired by rote memorizing. It is important then that the student understand each step that he takes in solving a problem. It is not sufficient that he understand the explanation given by the teacher but that he be able to give that explanation understandingly in his own words. In striving to attain this goal it would seem that short cuts, trick methods, catch phrases* and similar devices should be deleted from our teaching. The student who thoroughly understands his work will develop such devices for himself and having developed them will retain his understanding of the underlying principles. If the device is presented by the teacher the student, expecially the weaker one, will grasp it eagerly even as an adult grasps a relief dole to avoid the responsibility of holding a regular job. It is human nature to accept the easier way now rather than to be foresighted and to use the method which will provide the greatest ultimate benefit.

The second suggestion depends also upon the hypothesis that understanding promotes retention. This suggestion is that the student should learn the correct terminology from a teacher who uses the correct terminology. No one can profess perfection in this matter but students who misuse such common words as 'term', 'factor', 'power', 'exponent' and many others betray slovenliness in their teaching and reflect similar slovenliness in their thinking.

The third suggestion is based upon the hypothesis that repetition promotes retention. Would not a greater understanding and a greater retention be obtained if larger bodies of material were studied as units? First, survey the entire body of material, sketchily perhaps, but with careful regard for the understanding of the entire pattern and the dependence on the initial postulates. Second, repeat this study of the material several times, going more deeply into the subject matter and applications at each repetition. This constant repetition of the content without loss of sight of the unity of the subject matter should prevent the tendency to 'fail to see the forest because of the trees'.

*"Cross-multiply", "transpose and change sign" et cetera.

The fourth suggestion again depends upon the value of repetition. Mathematics should be taught and used in other courses just as correct grammar, diction, history, et cetera should be a part of the teaching of mathematics. The use of mathematics in the sciences, the arts and elsewhere, always the same principles but with ever new situations, will steadily strengthen the student's understanding of both mathematics and the fields in which he applies it.

The fifth suggestion has to do primarily with the colleges. The schools of higher learning might well look to their requirements for entrance and see perhaps that they are making certain formal requirements met by large numbers of students with widely divergent preparations. Wouldn't it be desirable to remove these formal requirements and to replace them with some form of entrance examination? These examinations would be used to determine which students are really prepared for the college courses upon which they are about to enter and which ones need to take non-credit, pre-college courses before they attempt their college work. Many of our failures among poorly prepared college freshmen might thus be avoided. At the same time the work in the present college courses could be raised in caliber to the place where it would be a stimulus rather than a drudgery to those students whose preparation is adequate. Many of our students in college falter in their sophomore year because the work has been so elementary in the first year that they have become seriously delinquent in their study habits.

The sixth suggestion is perhaps the basic one. It is, that the quality of the teachers, both high school and college, must be raised. We should study more, read widely, acquire at least a basic training in philosophy, logic, general science and those other courses to which mathematics is so closely related. Teachers of other subjects should be urged to *understand* and use mathematics not only as a tool but as a thought process.

Lastly let us not forget that perfection is not obtained by the waving of a wand but by the wielding of a hoe. A slow steady process of recognizance of weaknesses and the eradication of these weaknesses is an excellent assurance of healthy growth.

Grades in Freshman Algebra as Indicative of Later Success in Engineering Mathematics Courses*

By CHESTER C. CAMP
University of Nebraska

By making a statistical study of the future success or failure of students in algebra one might expect to determine whether this prerequisite course was properly taught or properly mastered. From this point of view honest-to-goodness prerequisite courses enjoy properties not possessed by non-prerequisite courses. A similar situation obtains in science, in foreign language, and in other fields.

This paper shows the results of a statistical study of the records of more than a thousand students over a period of six years. Although most of the students were in the College of Engineering, some were in the Arts and Science College. It is believed that the results would be similar for students in the other colleges.

The question arose whether the prerequisite course should be given special consideration. Inquiry was made whether a student should be allowed to go on to the next course in case he received the minimum passing grade, which is 60 at the University of Nebraska, or whether he should be required to have a grade of 70 or more. There is a rule at this University which requires four fifths of the semester hours offered for graduation to be of grade 70 or higher. This may necessitate repetition of certain courses graded in the sixties.

In view of this situation it seemed important to determine whether the mortality in future work would be appreciably lessened if credit in prerequisite courses were withheld for any mark under 70. It seemed quite all right in case a student decided not to take a course for which algebra was a prerequisite that he be allowed credit for any mark from 60 to 69 in algebra.

One might think that grades in mathematics would be remarkably consistent. If a student gets 90 in algebra he might be expected to receive high marks in future courses. The record shows, however, that he may have a grade in the sixties in the next course. A grade in the sixties may also be followed by a grade in the nineties. Nevertheless a

*Presented on May 11, 1940, before the Nebraska section of the Mathematical Association of America.

significant correlation of grades in algebra with those in the next course, for students who advanced as far as a first course in calculus, was actually found to exist. In fact $r=.58$ where r is the correlation coefficient. The next course was trigonometry or a combination of trigonometry and analytic geometry. The records for this two-way frequency distribution of some 552 students are grouped in Table I below. In Table II are grouped the records of students in algebra and their first course in calculus. Here the correlation coefficient was computed and found to be $r=.50$. These results are high enough to represent a significant relation.

TABLE I

Grade in Algebra

		65	75	85	95	Totals
Grades in following course	95	2	4	41	104	151
	85	7	35	76	59	177
	75	17	49	50	16	132
	65	17	35	19	3	74
	*	8	6	3	1	18
Total						552

TABLE II

Grade in Algebra

		65	75	85	95	Totals
Grades in calculus	95	2	3	23	73	101
	85	5	21	61	62	149
	75	9	35	56	25	125
	65	21	40	18	20	99
	*	14	19	16	5	54
Total						528

*Refers to the grades Incomplete, Conditioned, Dropped deliquent, and Failure.

It is of interest in passing to note that the average grade in calculus in Table II is about 4 points lower than that for the course immediately following algebra as found in Table I.

If one takes those students who had no grade less than 70 in subsequent courses he finds that eight students who were ranked in the sixties in algebra had no trouble in future courses, i. e. no grade less than 70. This shows that a change in the passing mark would militate against these eight. They were among fifty who were in the same algebra grade group as shown in Table III below:

TABLE III

	Grade in Algebra 90-100	80-89	70-79	60-69	Totals
Subsequent grades all $\geqq 70$	156	115	39	8	318
Subsequent grades all $\geqq 60$	21	39	42	23	125
1 repetition due to *	4	13	17	8	42
2 repetitions due to *			3	1	4
* in last calculus course	5	16	27	10	58
Totals	186	183	128	50	547
Proportion of students with 1 or more future grades < 70 in each algebra grade group	1/6	1/3	2/3	5/6	

Of the group of 186 getting grades in the nineties in algebra just one sixth got grades below 70 in one or more subsequent courses. Of 183 in the eighties in algebra about one third had trouble; of 128 in the seventies more than two thirds fell down in later courses; while of 50 in the sixties in algebra five sixths had difficulties.

It must be emphasized that no grade in algebra guarantees complete success in future courses. If a grade of 70 were required one must notice that nearly the same relative number of students would have difficulty as now, in fact more might have to repeat courses because eight of the fifty who got grades in the sixties would be added to the repeaters in algebra. The big difference in the proportions is not between the 70-79 and 60-69 grade groups but between the 80-89 and 70-79 grade groups.

There were 460 students who did not go as far as the calculus. Those who passed algebra were divided into four grade groups in algebra as in Table III. Among the students of the 70-79 grade group who took more mathematics 71 per cent had one or more subsequent grades <70. For those who had algebra grades in the 60-69 grade group the corresponding figure was 83 per cent. These proportions correspond to the ratios $^2/_3$ and $^5/_6$ in Table III, for students who took calculus. It really means in round numbers only seven tenths and eight tenths. Furthermore one would have to justify penalizing the one sixth of the 60-69 grade group who get along very well in future courses. Consequently one is led to the conclusion that these proportions for either category do not differ sufficiently to justify withholding credit in prerequisite courses from students in the 60-69 grade group.

Mathematical World News

Edited by
L. J. ADAMS

"The War Preparedness Committee of the American Mathematical Society and Mathematical Association of America would be aided in its work if mathematicians who are drafted or volunteer or otherwise enter the military or naval service of the United States, would at once notify the Committee of this fact. Recommendations as to the best use of specially qualified mathematicians can then be made. In writing to the Chairman of the Committee, please state age, and describe academic training, special scientific interests, academic, industrial, engineering or military experience, and give other pertinent facts." The foregoing is a copy of a notice from Professor Marston Morse, Chairman of the War Preparedness Committee, Fuld Hall, Princeton, N. J.

Scripta Mathematica announces that *The Spider as Architect and Mathematician* by D. E. Smith and Jekuthiel Ginsburg, is in course of preparation.

The Algebra of Omar Khayyam, by Dr. Daoud S. Kasir, is a publication of *The Mathematics Teacher,* 525 West 120th Street, New York City. The price is fifty cents.

Dr. R. G. D. Richardson, Brown University, has retired as secretary of the American Mathematical Society after twenty years of service in this position. The next volume of the *Bulletin* of the A. M. S. will be dedicated to Professor Richardson.

Professor J. W. Branson, head of the mathematics department at New Mexico College of Agriculture and Mechanics Arts, has been appointed dean of instruction.

The National Research Council announces the publication of *A Bibliography on Orthogonal Polynomials,* a report by a committee, of which Professor J. A. Shohat is chairman.

Graduate courses in mathematics to be offered at the University of California at Los Angeles during the coming summer session are:

1. *Partial Differential Equations of Mathematical Physics.* Dr. Angus E. Taylor.
2. *The Calculus of Variations.* Professor William M. Whyburn.

The University of Chicago Press announces a distribution sale of several hundred of their publications. This is in connection with the celebration of the fiftieth anniversity of the University of Chicago. There are four titles in the mathematics offerings:

1. *Finite Collineation Groups.* Hans F. Blichfeldt.
2. *Contributions to the Calculus of Variations, 1931-32.* Edited by Gilbert A. Bliss.
3. *Contributions to the Calculus of Variations, 1933-37.* Edited by Gilbert A. Bliss, L. M. Graves and W. T. Reid.
4. *Correlated Mathematics for Junior College.* Ernst R. Breslich.

Dr. Elisha S. Loomis, mathematics professor emeritus at Baldwin-Wallace College, died on December 11, 1940. Dr. Loomis was the author of several mathematics textbooks. His best known work is the *Pythagorean Proposition,* containing 120 possible proofs of this theorem.

Dr. Josef E. Hofmann, of Nordlingen, Germany, has been called to a professorship for the history of mathematics in the University of Berlin. The Royal Academy of Sciences has appointed him to the editorship of Leibniz' Collected Works.

The Virginia Polytechnic Institute will set up special short courses in several engineering fields, in order to help meet the shortage in technical men for many defense industries. Proposed courses include:

1. Explosives Production and Inspection.
2. Surveying and Topographic Mapping.
3. Machine Design.
4. Materials Inspection and Testing.
5. Marine Engineering.
6. Production Engineering.

Whether these or other courses will be given will depend upon whether there is sufficient interest to justify them. Further information may be secured from the Dean of Engineering, V. P. I., Blacksburg, Virginia.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

No. 206. Proposed by *George Dantzig*, University of Michigan.

Consider three circles in parallel planes. A cone (in fact, two cones) can be made to pass through two of the circles. Show that by a proper selection the three vertices of the cones passing through the pairs of circles will lie on a straight line. (This is a generalization of No. 159.)

Solution by the *Editors*.

If two parallel circles, with centers P and Q and radii a and b, are drawn on a cone, it is easy to see from similar triangles that the vertex of the cone lies on PQ and divides it, internally or externally, into segments whose ratio is a/b. (Let R be the center and c the radius of the third circle.) Then the various vertices of the various cones described in the problem are the points which divide the corresponding sides of the triangle PQR into segments whose ratios are $\pm a/b$, $\pm b/c$, $\pm c/a$. Since, with proper choice of sign, the product of these ratios is -1, it follows according to the converse of Menelaus' theorem that the points of division are collinear.

No. 344. Proposed by *C. Bull Foster*, Evanston, Illinois.

Prove: If $\{a_n\}$ is a sequence of complex terms such that

$$\frac{a_n}{a_{n+1}}=1+\mu/n+O(1/n^\lambda),\quad \lambda>1,\quad \mu=a+ib,$$

where a, b, λ are real and $a>0$, then

$$\sum (a_n - {}_pC_1 a_{n+1} + {}_pC_2 a_{n+2} - \cdots \pm a_{n+p})$$

is absolutely convergent for $p=1, 2, 3, \cdots$.

Solution by the *Proposer*.

It is known that, under the above hypothesis,

$$\sum (a_n - a_{n+1})$$

is absolutely convergent.* We may write

$$\sum (a_n - a_{n+1}) - \sum (a_{n+1} - a_{n+2}) = \sum (a_n - 2a_{n+1} + a_{n+2})$$

which, being the difference of absolutely convergent series, will converge absolutely. Assuming that the proposed result is true for the index p, we have similarly the absolutely convergent series

$$\begin{aligned} &\sum (a_n - {}_{p+1}C_1 a_{n+1} + {}_{p+1}C_2 a_{n+2} - \cdots \pm a_{n+p+1}) \\ &= \sum (a_n - {}_pC_1 a_{n+1} + {}_pC_2 a_{n+2} - \cdots \pm a_{n+p}) \\ &- \sum (a_{n+1} - {}_pC_1 a_{n+2} + {}_pC_2 a_{n+3} - \cdots \pm a_{n+p+1}) \end{aligned}$$

because ${}_pC_k + {}_pC_{k-1} = {}_{p+1}C_k$. Thus the proposition is established by induction.

No. 361. Proposed by *E. P. Starke*, Rutgers University.

Find all solutions of the equation, $2^x - 3^y = \pm 1$.

Solution by the *Proposer*.

Let x be any real number, positive if the upper sign is chosen, then we have obviously $y = \log(2^x \mp 1)/\log 3$.

For such a problem, rational solutions are usually demanded. The only rational solutions here are

$$2^1 - 3^0 = 1, \quad 2^2 - 3^1 = 1, \quad 2^1 - 3^1 = -1, \quad 2^3 - 3^2 = -1,$$

It is easily seen that x and y are non-negative integers. For $2^x - 3^y = 1$ we have $x=1$, $y=0$ and $x=2$, $y=1$. But if $x \geqq 3$, then

$$3^y \equiv -1 \bmod 8$$

which is easily seen to be impossible. For $3^y - 2^x = 1$ we have $x=1$, $y=1$ and $x=3$, $y=2$. $x \geqq 3$ implies $3^y \equiv 1 \bmod 8$, whence y is even. Put $y=2v$ to get $(3^v-1)(3^v+1) = 2^x$. Then $3^v - 1 = 2^h$, $3^v + 1 = 2^k$,

*Bromwich, *Theory of Infinite Series*, Second Edition, 1926, p. 241, where it is shown also that for $0 < a \leqq 1$. Σa_n diverges although $a_n \to 0$.

$h+k=x$. Thus $2^h+2=2^k$ and so $k>h$, but comparison with the identity $2^h+2^h=2^{h+1}$ shows that $k<h+1$ unless $h=1$; but then $h=1$, $k=2$, $x=3$.

No. 363. Proposed by *Howard D. Grossman*, New York City.

The following columns of numbers (which may be indefinitely extended) are used in the old, well known trick of finding a number (as an age), given only the columns in which it appears.

1	2	4	8
3	3	5	9
5	6	6	10
7	7	7	11
9	10	12	12
11	11	13	13
13	14	14	14
15	15	15	15

Each integer may be expressed in just one way as a sum of powers of 2. Every integer in these columns is written once in each column headed by a power of 2 which appears in the sum for this integer. The integers in each column are written downward in order of increasing magnitude.

Now in each row there are four integers, some different, some alike. The integers in any row may be separated into sets, each set containing repetitions of the same integer. Let n_1 be the number of terms in the first set, n_2 the number in the second set, etc. Then all the sets $n_1, n_2, n_3, \cdots$ for all the rows constitute all the different partitions (including their permutations) of the number of columns, 4, into positive integers, viz., 1111, 211, 121, 31, 112, 22, 13, 4.

Prove that this is generally true regardless of the number of columns in the table.

Solution by the *Editors*.

Starting with a table, T_n, of n columns, and assuming that $T_i(i=1,2,\cdots,n)$ satisfy all the statements of the proposal, we shall build up a table, T_{n+1}, of $n+1$ columns. (T_n consists of the numbers 1 to 2^n-1, inclusive, each written one or more times, and comprises just 2^{n-1} rows). The numbers 2^n+A, $A=0, 1,\cdots, 2^{n-1}-1$, are written in order in the $(n+1)$th column opposite the 2^{n-1} rows of T_n; they also appear in the first $n-1$ columns of 2^{n-2} new rows in just the positions that the numbers A occupy in the first 2^{n-2} rows (constituting thus essentially T_{n-1}). The next 2^{n-2} numbers, $2^n+2^{n-1}+B$, $B=0, 1,\cdots, 2^{n-2}-1$, are written in order in both the nth and $(n+1)$th columns opposite T_{n-1}, and also in the first $n-2$ columns of 2^{n-3} new rows where they constitute T_{n-2} (except for the addition of 2^n+2^{n-1} to each entry). The last three columns opposite T_{n-2} are filled in triplicate with the 2^{n-3} numbers,

$$2^n+2^{n-1}+2^{n-2}+C,\ C=0, 1,\cdots, 2^{n-3}-1,$$

which then give rise to a T_{n-3} in 2^{n-4} new rows, etc. Finally in a 2^nth row, the single entry $2^n+2^{n-1}+2^{n-2}+\cdots+2+1(=2^{n+1}-1)$ appears in every column.

Thus the first 2^{n-1} rows of T_{n+1} present all the partitions of n with an additional 1 at the right. The next 2^{n-2} rows present all the partitions of $n-1$ with an additional 2 at the right. Continuing in this way we have for each $k(k=1, 2,\cdots, n)$ all the partitions of $n+1-k$ with an additional k; and finally in the 2^nth row the partition $(n+1)$. Since this evidently presents all the partitions of $n+1$, we have a proof by induction of the assumed properties.

No. 366. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

The polar of the vertex A with respect to the variable circle through the vertices B and C of triangle ABC meets the circle in the points P and Q. The perpendicular from A upon this polar meets the circle in the points R and S.

1. P and Q trace the cubic curve T_1.
2. R and S trace the cubic curve T_2 which is orthogonal to T_1 at the vertices of ABC.
3. T_1 is tangent to the circumcircle of ABC at A. T_2 is tangent at A to the Appolonian circle of ABC passing through A.
4. The center of curvature of T_1 at A is the midpoint of the segment joining the circumcenter to A.
5. The center of curvature of T_2 at A is the midpoint of the exsymmedian of ABC issued from A.

Solution by the *Proposer*.

Let the triangle have vertices $A(f,g)$, $B(0,c^{1/2})$, $C(0,-c^{1/2})$. The equation of a variable circle through B and C is

$$(1) \qquad x^2+y^2+2ax-c=0. \quad (a \text{ a parameter}).$$

The polar of A with respect to (1) is

$$(2) \qquad x(a+f)+yg+af-c=0.$$

The perpendicular from A upon (2) is

$$(3) \qquad gx-(a+f)y+ag=0.$$

1. The locus of P and Q is found by eliminating a between (1) and (2).

$$T_1 : x^3+xy^2-fx^2-2gxy+fy^2+cx-cf=0.$$

2. The locus of R and S is found by eliminating a between (1) and (3).

$$T_2 : y^3+yx^2+gx^2-2fxy-gy^2-cy+cg=0.$$

Differentiate T_1 and T_2 respectively with respect to x to find for

$$T_1 : (4)\ y'=(2gy+2fx-y^2-3x^2-c)/(2xy-2gx-2fy)$$

and for $T_2 : (5)\ y'=(2fy-2xy-2gx)/(3y^2+x^2-2fx-2gy-c).$

If these are evaluated at the points A,B,C it will be found they are respectively negative reciprocals, and hence T_1 and T_2 are orthogonal at the vertices of ABC.

3. The circumcircle of ABC is

$$f(x^2+y^2)-(f^2+g^2-c)x-fc=0. \tag{6}$$

Differentiate (6) with respect to x to find

$$y'=(f^2+g^2-2fx-c)/2fy. \tag{7}$$

If (4) and (7) are evaluated at A, both reduce to $y'=(g^2-f^2-c)/2fg$. The Appolonian circle of ABC through A is

(8) $\qquad g(x^2+y^2)-(g^2+f^2+c)y+cg=0.$ (See Prob. 333, page 41, Oct.)

Differentiate (8) with respect to x to get

$$y'=-2gx/(2gy-g^2-f^2-c). \tag{9}$$

If (5) and (9) are evaluated at A, both reduce to $y'=-2fg/(g^2-f^2-c)$.

4. The circumcenter of ABC is $O[(f^2+g^2-c)/2f,0]$. The midpoint of the segment OA is $M[(3f^2+g^2-c)/4f,g/2]$. Differentiate (4) with respect to x to get

$$y''=[2(g-y)y'-(x+f)y'^2+f-3x]/(xy-gx+fy).$$

At $A(f,g)$, $\quad y''=-2(1+y'^2)/g$ and $y'=(g^2-f^2-c)/2fg$.

These values placed in the standard formulas

$$X=x-y'(1+y'^2)/y'', \quad Y=y+(1+y'^2)/y'' \tag{10}$$

give M as the center of curvature of T_1 at A.

5. The center of (8) is $N[0, (g^2+f^2+c)/2g]$ and is the intercept of the exsymmedian issued from A upon the side BC of ABC. The midpoint of AN is $L[f/2, (3g^2+f^2+c)/4g]$. Differentiate (5) with respect to x to get

$$y''=2[2y'(f-x)-y(3y'^2+1)-g(y'^2+1)]/(3y^2+x^2-2fx-2gy-c).$$

At $A(f,g)$ $y''=-4g(y'^2+1)/(g^2-f^2-c)$ and $y'=-2fg/(g^2-f^2-c)$.

These values placed in formulas (10) give L to be the center of curvature of T_2 at A.

Note from 3. We have incidentally proved the well known theorem: an Appolonian circle is orthogonal to the circumcircle at the vertex through which the Appolonian circle passes.

PROPOSALS

No. 387. Proposed by *W. V. Parker*, Louisiana State University.

If the curve $y=x^4+ax^3+bx^2+cx+d$ has two points of inflection and intercepts segments of lengths d_1, d_2, d_3 in order on the line joining them show that

$$4d_1=4d_3=(\sqrt{5}-1)d_2.$$

No. 388. Proposed by *Harold S. Grant*, Rutgers University.

Show how to obtain the necessary and sufficient conditions that a polynomial equation of degree n possess a root of multiplicity $m(2<m\leqq n)$ in terms of rational, integral functions of the coefficients.

No. 389. Proposed by *E. P. Starke*, Rutgers University.

Tangents to the parabola $y^2=4ax$ at any two points A, B meet in a point whose ordinate is the arithmetic mean of the ordinates of A and B and whose abscissa is a geometric mean of the abscissas of A and B.

No. 390. Proposed by *E. P. Starke*, Rutgers University.

Find the number of triangles of all kinds whose sides are positive integers and whose largest side does not exceed a given number, K.

No. 391. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

According to Louis C. Karpinski, the following theorem, ascribed to Archimedes by Albiruni (c. 1,000 A. D.), was recently brought to light in connection with an Arabic work on trigonometry and is now believed to have been the basis of Greek trigonometry before Ptolemy (150 A. D.)

Given chords AB and BC, $AB>BC$, prove that if M is the midpoint of arc ABC, then MN, perpendicular to AB at N, bisects the broken line ABC.

No. 392. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given the circumcenter O and the ex-centers O_a, O_b, construct the triangle ABC.

No. 393. Proposed by *Dewey C. Duncan*, Los Angeles City College.

Let the proper fraction a/b be reduced to a repeating decimal fraction whose period P consists of n digits. State and prove the general theorem of which the following are special cases:

(a) If $n=1$, then 3 is a divisor of b.

(b) If $n>1$, and 3 is not a divisor of b, then 3 is a divisor of P.

(c) If $n=3$ and 37 is not a divisor of b, then 37 is a divisor of P.

No. 394. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

If P, Q are points on $x^2/a^2+y^2/b^2=1$ whose eccentric angles Θ_1 and Θ_2 satisfy the relation $(1/2)\sin(\Theta_1+\Theta_2)=\cos\frac{1}{2}(\Theta_1-\Theta_2)$, prove that PQ envelopes the curve $x^{2/3}/a^{2/3}+y^{2/3}/b^{2/3}=1$.

395. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Prove that the locus of the centers of all conicoids through the cubic curve $x=t$, $y=t^2$, $z=t^3$ is the surface $2x^3-3xy+z=0$, $(x=u$, $y=v$, $z=3uv-2u^3)$. Find the asymptotic lines of this surface and show that their projections on the plane $x=0$ envelope the semicubical parabola $z^2=2y^3$.

No. 396. Proposed by *N. A. Court*, University of Oklahoma.

The six orthogonal projections of a vertex of a tetrahedron upon the internal and external bisecting planes of the three dihedral angles which the face opposite the vertex considered makes with the remaining faces, are coplanar.

Bibliography and Reviews

Edited by
H. A. SIMMONS

Living Mathematics. By Ralph S. Underwood and Fred W. Sparks. New York, McGraw-Hill Book Co., Inc., 1940, 9+365 pp.

This is another interesting attempt to provide a textbook for those students who take only one year of mathematics in college. The major part of the book deals with algebra, which is supplemented by a survey of trigonometry, analytic geometry, calculus, and theory of numbers. According to the authors the aims are to "provide enough of the conventional subject matter to meet practical credit transfer requirements", to give "for non-specialists the interest that is inherent in mathematics itself", and to foster "an appreciation of its place in modern life."

The review and extension of the algebraic principles include the usual topics with the happy inclusion of interest and annuity problems, following progressions. In the elementary theory of equations, we find the graphical method used to approximate the irrational roots of algebraic and transcendental equations. Unfortunately an introduction to statistics has been omitted.

In trigonometry, the definitions of the trigonometric functions for acute angles, and then for general angles, are given. The solution of triangles and some work with identities involving one angle complete this phase of the book. The usual introductory topics in analytic geometry precede the derivation of equations of the standard curves. The simple type equations of the conic sections are derived, using the eccentricity definitions, and then the axes are translated. This is a sensible time saver. Finally, the derivative arises in a discussion of velocity, but the slope interpretation fixes the ideas of the "delta method" of differentiation, which is used only for algebraic functions. The ten pages of integral calculus may be difficult to teach but it is not bad to include them.

The book is written in a pleasing, conversational tone, with hovel headings for the articles. It would improve the book to add appropriate subtitles, since, for example, "Surprising mixups" does not suggest permutations and combinations. In the index one can find the usual words, but the table of contents is useless.

The discussions and explanations are adequate and indeed stimulating. Applications are fairly numerous, though there should be still more of them. The authors have achieved their purpose.

Oberlin College. M. M. JOHNSON

Plane Trigonometry. Revised edition. By Raymond W. Brink. D. Appleton-Century Company, Inc., New York, 1940. xii+226+110 pages.

As the author states in the preface, "This book is a complete revision of the author's Plane Trigonometry". The numerous illustrative examples are mostly new and well chosen. The exercises are adequate and well graded. In general, the topics are treated in detail and with some repetitions, so that the book is quite long. However, by the omission of certain indicated topics and some contractions, the text could be adapted

for use in the usual two hour semester course. Answers to most of the odd numbered problems are given, but the reviewer has not had time to check any of them.

After defining the trigonometric functions of a general angle, the author confines the treatment to acute angles and the solution of right triangles by means of logarithms and tables before proceeding with the general concepts. Inverse trigonometric functions and trigonometric equations are introduced early and are repeated from time to time with increasing generality and difficulty. The definitions of the principal inverse trigonometric functions are not, in all cases, those which are found to be convenient in the Calculus. They are simpler than the usual definitions but are subject to the criticism of discontinuity in the arcsecant and arccosecant functions. The reviewer is pleased to see that some advanced topics are included, but he believes that the author's attempt to derive a series expansion and his operations upon such expansions are very misleading.

Instances of vagueness and inaccuracy have been found, some of which it might be well to list here. The word "number" is used without sufficient qualification in the chapter on logarithms. The author fails to distinguish between definitions and "laws" of exponents. The formulas $\tan x/2 = \sin x/(1+\cos x)$ and $\tan x/2 = (1-\cos x)/\sin x$ are not listed except as problem identities on page 148 where we find the first listed correctly and also with a $\pm$ sign, while the latter is listed with a $\pm$ sign in a slightly different form.

The above criticisms do not detract measurably from the good typography and arrangement of the text. Most instructors should find it adequate for either a long or a brief course in Plane Trigonometry.

Northwestern University. J. M. DOBBIE.

The Nineteen Forty Mental Measurements Yearbook. By Oscar K. Buros. Rutgers University Press, Highland Park, N. J., 1941. xxi+674 pages.

The latest edition of the *Mental Measurements Yearbook* is the handiwork of Oscar K. Buros of Rutgers University and of some two hundred fifty psychologists, subject-matter specialists, teachers, and test technicians of the United States, Canada, England and Scotland, who have contributed reviews. Starting in 1935 with a 44 page paper-bound booklet, the *Yearbook* now has become an attractively bound volume of 674 double-column pages.

The first major section is devoted to *Tests and Reviews* with sub-sections on "achievement batteries", "character and personality", "English", "fine arts", "foreign languages", "intelligence", "mathematics", "reading", "science", "social studies", "vocations", and another headed "miscellaneous". The purposes of this section, as stated by the editor, are as follows: "(a) to make readily available comprehensive and up-to-date bibliographies of recent tests published in all English-speaking countries; (b) to make readily available hundreds of frankly critical test reviews, written by persons of outstanding ability representing various viewpoints, which will assist test users to make more discriminating selections of the standard tests which will best meet their needs; (c) to make readily available comprehensive and accurate bibliographies of references on the construction, validation, use, and limitations of specific tests; (d) to impel authors and publishers to place fewer but better tests on the market and to provide test users with detailed and accurate information on the construction, validation, uses, and limitations of their tests at the time that they are first placed on the market; (e) to suggest to test users better methods of arriving at their own appraisals of both standard and non-standard tests in light of their particular

values and needs; (f) to stimulate co-operating reviewers—and others to a less extent—to reconsider and think through more carefully their beliefs and values relevant to testing; (g) to inculcate upon test users a keener awareness of both the values and dangers which may accompany the use of standard tests; and (h) to impress test users with the desirability of suspecting all standard tests—even though prepared by well-known authorities— unaccompanied by detailed data on their construction, validation, use, and limitations."*

It will interest mathematicians to know that 47 pages are devoted in this first section to the field of mathematics, including the listing of 44 instruments or batteries of tests, reviews of these tests, and a list of published reports of studies involving the use or the validation of each. The attempt has been made to list tests made available to the public as late as October 1, 1940.

A second large section entitled "Books and Reviews" has the following avowed purposes: "(a) to make readily available comprehensive and up-to-date bibliographies of recent books published in all English-speaking countries in measurements and closely associated fields; (b) to make readily available evaluative excerpts from hundreds of book reviews appearing in a great variety of journals in this country and abroad in order to assist test users to make more discriminating selections of books for study and purchase; (c) to stimulate readers to develop more critical attitudes toward what they read; (d) to make readily available important and provocative statements which, though appearing in book reviews, have considerable value entirely apart from a consideration of the book under review; (e) to point out books which are not being reviewed but which probably merit review; (f) to improve the quality of book reviews by stimulating review editors to make greater efforts to choose competent reviewers who will contribute frankly critical reviews; and finally, (g) to improve the quality of book reviews by stimulating reviewers 'to take their responsiblities more seriously' by refusing to review books which they cannot appraise competently and honestly."†

The 170 pages of titles and quoted reviews (in most instances reviews from American and British periodicals, but books gathered from as far away as India and South Africa) include bibliographies, textbooks in the field of measurements, reports of testing programs, volumes dealing with diagnosis and evaluation in the various curricular fields, and a series of publications dealing with miscellaneous topics, including "character and personality testing", "intelligence testing", "vocations and guidance", "civil service examinations", "records and reports", "observational methods".

These major sections are followed by a directory and index of periodicals and of the review editors of each periodical, and a directory and index of publishers. The section on research and statistical books which was included in the 1938 edition is omitted, but this section is to be published as a separate volume in the near future.

This *Yearbook* seems superior to other volumes in the series because it has a greater number of reviews, because an effort has been made to have these reviews written by critics who differ in their opinions of each test, and because the reviews are longer and give a more complete analysis than was possible in previous editions. The volume gives evidence of the same thoroughness and painstaking efforts which have characterized the work of Dr. Buros in the past. It is no less severe in its criticism of tests than was the last edition which caused much condemnation to be directed toward the editor. We are pleased to observe this adherence to frank criticism; for it is the one characteristic which would seem most necessary if the volume is to serve its declared purpose. In the selection of reviewers, every effort has apparently been made

*p. xvii.
†p. xix.

to avoid bias,—either bias due to personal animosity or to strong personal friendship toward the authors of tests and publications. Dr. Buros has made an important contribution to professional literature.

Northwestern University. KENNETH L. HEATON.

Applied Mathematics in Chemical Engineering. By Thomas K. Sherwood and Charles E. Reed. McGraw-Hill, New York, 1939. xi+403 pages. $4.00.

The table of contents of *Applied Mathematics*, briefly annotated by the reviewer, will serve to show the differences between this text and the usual texts in advanced calculus and in applied mathematics. Integration and Differentiation (including both graphical and numerical methods), The Use of Differential Equations (treating the problem of setting up a wide variety of chemical engineering problems in the form of ordinary differential equations), Solution of Ordinary Differential Equations (including both graphical and numerical methods), Applications of Partial Differential Equations, Infinite Series (including Fourier series and Bessel functions), Partial Differential Equations, Numerical Analysis (including curve fitting, interpolation, numerical methods for differentiation and integration), Graphical Treatment of Chemical Engineering Processes (including various special types of graph paper and their uses, alignment charts), Theory of Errors and Precision of Measurements (including the Theory of Least Squares).

The preface contains the following enlightening remarks:

> "...These mathematical techniques will assume an increasing importance in the future, and the engineer without an adequate mathematical background will find himself under an even greater handicap than at the present time. It is already evident that the undergraduate courses in differential and integral calculus are inadequate as a basis for following the current literature both in applied chemistry and in the unit operations. ... The usual course in advanced calculus includes many proofs of theorems which the student is willing to take for granted, and includes topics, such as vector analysis, for which he finds little application in his work. The courses and texts on differential equations are usually excellent in their handling of the problem of solving a differential equation, but of little help in the equally important problem of setting up a differential equation to express a physical or chemical problem. It is hoped that the present book, written by chemical engineers who do not pretend to be mathematicians, will prove helpful to those who have found books on applied mathematics written by mathematicians to be unsatisfactory, and that it may be of sufficient practical use to offset its deficiencies from the mathematician's point of view."

In answer to the prefatory remark concerning applied texts by mathematicians, the reviewer would call attention to two excellent texts that have been recently published: one by an applied mathematician and an electrical engineer (Mathematics of Modern Engineering by Doherty and Keller), the other by two applied mathematicians (Advanced Mathematics for Engineers by Reddick and Miller). It seems to the reviewer that such a co-authorship of men with special engineering background and with mathematical background is the ideal one for the writing of any text in applied mathematics.

Contrary to present usage the only problems for assignment are given in an appendix at the end of the book. These number seventy-two, thirty being routine mathe-

matical problems and the others engineering problems (mostly from chemical engineering) requiring the use of the mathematics of the text. For pedagogical purposes the grouping is awkward and the number inadequate. It is true, however, that the numerous illustrative examples, mostly from chemical engineering, admirably serve to show how the various mathematical topics are to be used. A study of this text should give the student ability to set-up such problems—a primary aim of the authors.

As would be expected from the statement quoted from the preface there are a few items to which the reviewer, as a teacher of mathematics, objects; the following examples illustrate these objections. On page 29 the authors refer to a general conic as the "most general parabola" and continue in this vein for the next paragraph. On page 144 is found the following statement with no qualifications: "It can be demonstrated with rigor that for any number of differentiations or variables the order of differentiation in the higher partial derivatives is immaterial." On pages 184-185 one finds correct statements for the comparison tests for convergence and divergence, but the illustrative example shows only the first five numerical terms with no mention of the general term.

In spite of these objectionable features (which could be corrected by the teacher), this text will fill a much needed place in the slowly growing list of applied mathematics textbooks for special groups of students. It is well written and the figures well drawn. This reviewer would welcome more texts, written by equally outstanding men in the other main branches of engineering and with the same general aims in mind.

North Carolina State College. JOHN W. CELL.